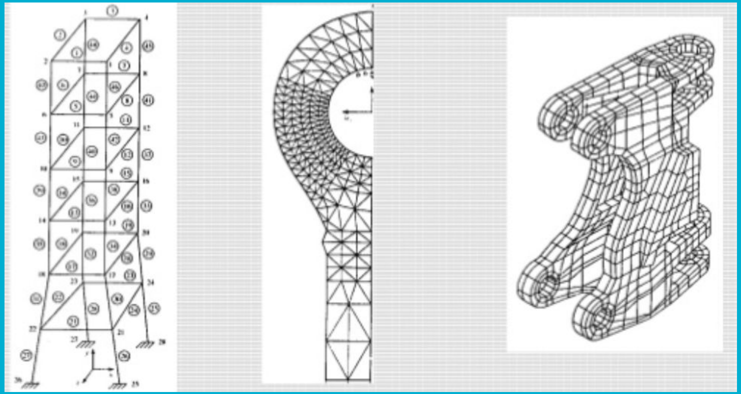
Finite Element Analysis



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FEA Interview

- FEA Theory
- Material Science
- Mathematics Matrix Algebra and Calculus
- Explicit and Implicit Schemes
- LS DYNA Theory
- Meshing Theory
- Crash Fundamentals

Science: Science is the sum of systematic human knowledge gained by man through observation and experimentation.

Engineering: Engineering is an human activity which is directed towards achieving better artifacts,processes,algorithms and systems which can serve humans.

Methods to Solve Any Engineering Problem

Analytical Method	Numerical Method	Experimental Method
• Classical approach • 100% accurate results • Closed form solution • Applicable only for simple problems like cantilever and simply supported beams, etc. • Complete in itself	• Mathematical representation • Approximate, assumptions made • Applicable even if a physical prototype is not available (initial design phase) • Real life complex problems • Results cannot be believed blindly. Certain results must be validated by experiments and/or analytical method.	• Actual measurement • Time consuming and needs expensive set up • Applicable only if physical prototype is available • Results cannot be believed blindly and a minimum of 3 to 5 prototypes must be tested
Though analytical methods could also give approximate results if the solution is not closed form, in general analytical methods are considered as closed form solutions i.e. 100% accurate.	Finite Element Method: Linear, nonlinear, buckling, thermal, dynamic, and fatigue analysis Boundary Element Method: Acoustics, NVH Finite Volume Method: CFD (Computational Fluid Dynamics) and Computational Electromagnetics Finite Difference Method: Thermal and Fluid flow analysis (in combination with FVM)	• Strain gauge • Photo elasticity • Vibration measurements • Sensors for temperature and pressure, etc. • Fatigue test

Finite Element Method (FEM):

FEM is the most popular numerical method.

The Finite Element Method (FEM) is a numerical technique used to determine the approximated solution for a partial differential equations (PDE) on a defined domain (W). To solve the PDE, the primary challenge is to create a function base that can approximate the solution. There are many ways of building the approximation base and how this is done is determined by the formulation selected. The Finite Element Method has a very good performance to solve partial differential equations over complex domains that can vary with time.

A geometric model becomes a mathematical model, when its behaviour is described, or approximated by selected differential equations and boundary conditions.

3 Methods to solve any engineering problem

Boundary Element Method (BEM):

This is a very powerful and efficient technique to solve acoustics or NVH problems. Just like the finite element method, it also requires nodes and elements, but as the name suggests it only considers the outer boundary of the domain. So, when the problem is of a volume, only the outer surfaces are considered. If the domain is of an area, then only the outer periphery is considered. This way it reduces the dimensionality of the problem by a degree of one and thus solving the problem faster.

The Boundary Element Method (BEM) is a numerical method of solving linear PDE which have been formulated as integral equations. The integral equation may be regarded as an exact solution of the governing partial differential equation. The BEM attempts to use the given boundary conditions to fit boundary values into the integral equation, rather than values throughout the space defined by a partial differential equation. Once this is done, in the post-processing stage, the integral equation can then be used again to calculate numerically the solution directly at any desired point in the interior of the solution domain. The boundary element method is often more efficient than other methods, including finite elements, in terms of computational resources for problems where there is a small surface/volume ratio. Conceptually, it works by constructing a “mesh” over the modeled surface. However, for many problems boundary element methods are significantly less efficient than volume-discretization methods like FDM, FVM or FEM.

Finite Volume Method (FVM):

The Finite Volume Method (FVM) is a method for representing and evaluating partial differential equations as algebraic equations [LeVeque, 2002; Toro, 1999]. It is very similar to FDM, where the values are calculated at discrete volumes on a generic geometry. In the FVM, volume integrals in a partial differential equation that contain a divergence term are converted to surface integrals, using the divergence theorem. These terms are then evaluated as fluxes at the surfaces of each finite volume. Because the flux entering a given volume is identical to that leaving the adjacent volume, these methods are conservative. Another advantage of the finite volume method is that it is easily formulated to allow for unstructured meshes. The method is used in many computational fluid dynamics packages.

Finite Difference Method (FDM):

Finite Element and Finite Difference Methods share many common things. In general, the Finite Difference Method is described as a way to solve differential equation. It uses Taylor’s series to convert a differential equation to an algebraic equation. In the conversion process, higher order terms are neglected. It is used in combination with BEM or FVM to solve thermal and CFD coupled problems.

FEM VS FDM

- FDM makes **pointwise approximation to the governing equations i.e. it ensures continuity only at** the node points. Continuity along the sides of grid lines are not ensured.

- FEM make piecewise approximation i.e. it ensures the continuity at node points as well as along the sides of the element.

- FDM do not give the values at any point except at node points. It do not give any approximating function to evaluate the basic values (deflections, in case of solid mechanics) using the nodal values.

Question: Is it possible to use all of the methods listed above (FEA, BEA, FVM, FDM) to solve the same problem (for example, a cantilever problem)?

The answer is YES! But the difference is in the accuracy achieved, programming ease, and the time required to obtain the solution.

When internal details are required (such as stresses inside the 3D object) BEM will lead to poor results (as it only considers the outer boundary), while FEM, FDM, or FVM are preferable. FVM has been used for solving stress problems but it is well suited for computational fluid dynamics problems where conservation and equilibrium is quite natural. FDM has limitations with complicated geometry, assembly of different material components, and the combination of various types of elements (1D, 2D and 3D). For these types of problems FEM is far ahead of its competitors.

Discretization of Problem:

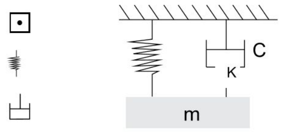
All real-life objects are continuous. This means there is no physical gap between any two consecutive particles. As per material science, any object is made up of small particles, particles of molecules, molecules of atoms, and so on and they are bonded together by the force of attraction. Solving a real-life problem with the continuous material approach is difficult. The basis of all numerical methods is to simplify the problem by discretizing (discontinuation) it. In other words, nodes work like atoms and the gap in between the nodes is filled by an entity called an element. Calculations are made at the nodes and results are interpolated for the elements.

From a mechanical engineering point of view, any component or system can be represented by three basic elements:

Mass 'm'

Spring 'k'

Damper 'c'



All the numerical methods including the Finite Element Method follow the discrete approach. Meshing (nodes and elements) is nothing but the discretization of a continuous system with infinite degrees of freedom to a finite degrees of freedom.

There are two approaches to solve any problem

Continuous approach

All real-life components are continous

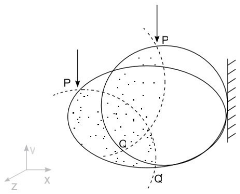
Discrete approach

Equivalent mathematical modeling

Discrete (mathematical equivalent) model, chair represented by shell and beam elements, person via lumped mass at C.G.

When Can We Say That We Know the Solution to The Above Problem?

If and only if we are able to define the deformed position of each and every particle completely.



The minimum number of parameters (motion, coordinates, temperature, etc.) required to define the position and state of any entity completely in space is known as degrees of freedom (dof)

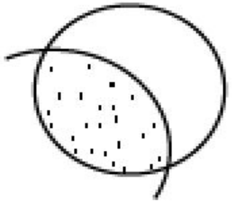
FEM VS Classical Methods

- In classical methods exact solutions are obtained where as in finite element analysis approximate solutions are obtained.
- Whenever the following complexities are faced, classical method makes the drastic assumptions' and looks for the solutions:
  - Shape, Boundary conditions and Loading
- When material property is not isotropic, solutions for the problems become very difficult in classical method but in FEM solutions for the problems without any difficulty.
- If structure consists of more than one material, it is difficult to use classical method, but finite element can be used without any difficulty.
- Problems with material and geometric non-linearities can not be handled by classical methods. There is no difficulty in FEM.

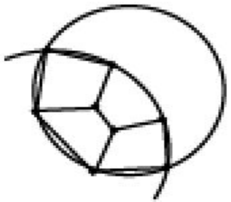
The total DOFs for a given mesh model is equal to the number of nodes multiplied by the number of dof per node.

All of the elements do not always have 6 dofs per node. The number of dofs depends on the type of element (1D, 2D, 3D), the family of element (thin shell, plane stress, plane strain, membrane, etc.), and the type of analysis. For example, for a structural analysis, a thin shell element has 6 dof/node (displacement unknown, 3 translations and 3 rotations) while the same element when used for thermal analysis has single dof /node (temperature unknown).

For a new user, it is a bit confusing but there is a lot of logical, engineering, and mathematical thinking behind assigning the specific number of dofs to different element types and families.



No. of points = ∞
DOF per point = 6
Total equations = ∞



No. of nodes = 8
DOF per node = 6
Total equations = 48

Why Do We Carry Out Meshing? What Is FEM / FEA?

FEM

- A numerical method
- Mathematical representation of an actual problem
- Approximate method

The Finite Element Method only makes calculations at a limited (Finite) number of points and then interpolates the results for the entire domain (surface or volume).

Finite – Any continuous object has infinite degrees of freedom and it is not possible to solve the problem in this format. The Finite Element Method reduces the degrees of freedom from infinite to finite with the help of discretization or meshing (nodes and elements).

Element – All of the calculations are made at a limited number of points known as nodes. The entity joining nodes and forming a specific shape such as quadrilateral or triangular is known as an Element. To get the value of a variable (say displacement) anywhere in between the calculation points, an interpolation function (as per the shape of the element) is used.

Method - There are 3 methods to solve any engineering problem. Finite element analysis belongs to the numerical method category.

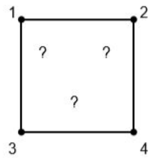
How the Results are Interpolated from a Few Calculation Points

It is ok that FEA is making all the calculations at a limited number of points, but the question is how it calculates values of the unknown somewhere in between the calculation points.

This is achieved by interpolation. Consider a 4 noded quadrilateral element as shown in the figure below. A “quad4” element uses the following linear interpolation formula:

$$u = a_0 + a_1 x + a_2 y + a_3 xy$$

FEA calculates the values at the outer nodes 1, 2, 3, 4 i.e. a_0, a_1, a_2, a_3 are known.

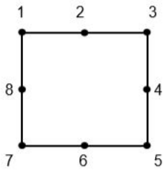


4 noded (linear) quad

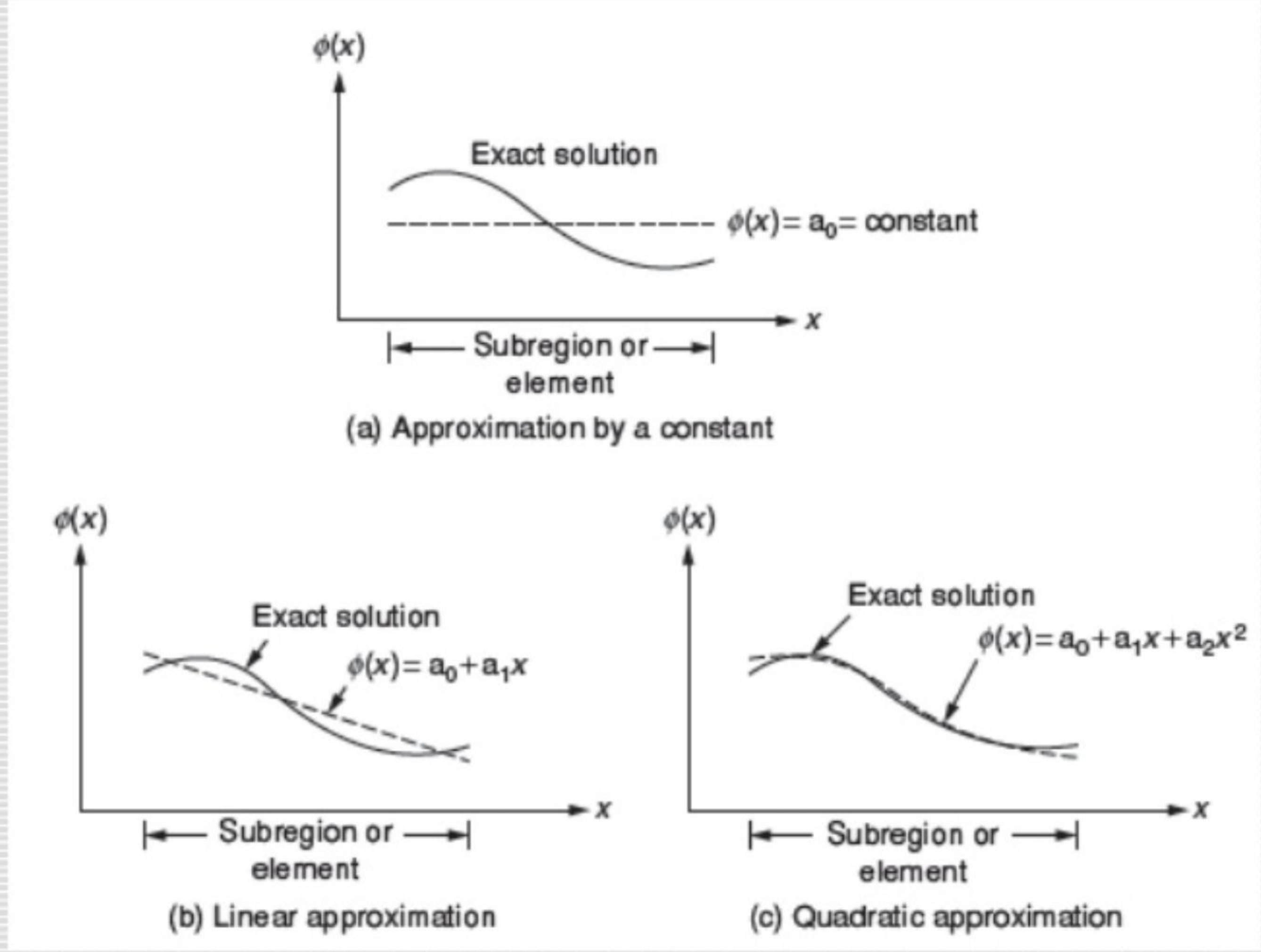
The value of the variable anywhere in between could be easily determined just by specifying x and y coordinates in above equation.

For an 8 noded quadrilateral, the following parabolic interpolation function is used:

$$u = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x_2 + a_5 y_2 + a_6 x_2 y + a_7 xy_2$$



8 noded (parabolic) quad

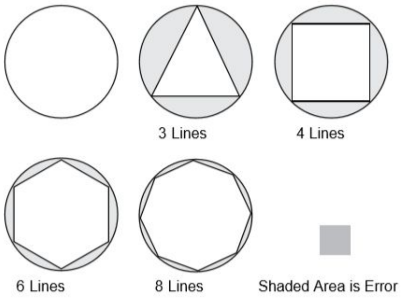


Higher Order Polynomial for shape functions give more accurate results

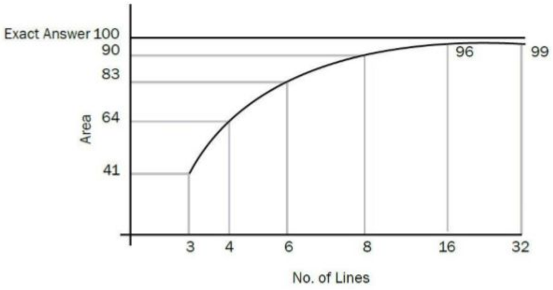
How is the Accuracy if we Increase the Number of Calculation Points (Nodes and Elements)?

In general, increasing the number of calculation points improves the accuracy.

Suppose somebody gives you 3 straight lines and asks you to best fit it in a circle, then find the area of the triangle and compare it with the circle area. This is then repeated with 4, 6, 8, 16, 32 and 64 lines.



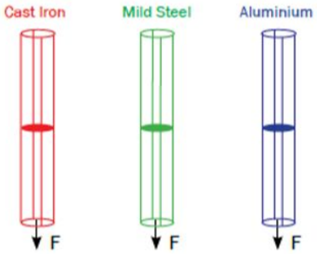
By increasing the number of lines, the error margin reduces. The number of straight lines is equivalent to the number of elements in Finite Element Analysis.



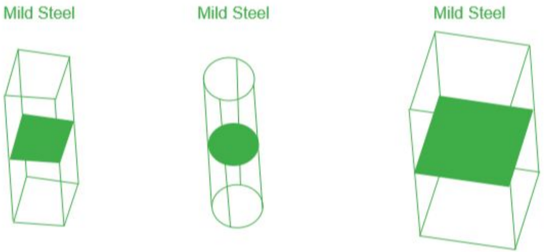
The exact answer for the area of the circle (πr^2) is 100. 3 lines gives the answer of 41, while 4 lines gives 64, and so on. An answer of 41 or 64 is not at all acceptable, but 80 or 90 is, considering the time spent and the relative design concept.

What is Stiffness and Why Do We Need it in FEA?

Stiffness 'K' is defined as Force/length (units N/mm). Physical interpretation – Stiffness is equal to the force required to produce a unit displacement. The stiffness depends on the geometry as well as the material properties.



Consider 3 rods of exactly the same geometrical dimensions – Cast Iron, Mild Steel, and Aluminium. If we measure the force required to produce a 1 mm displacement then the Cast Iron would require the maximum force, followed by Steel and Aluminium respectively, indicating $K_{CI} > K_{MS} > K_{Al}$



Now consider 3 different cross-sectional rods of the same material. Again, the force required to produce a unit deformation will be different. Therefore, stiffness depends on the geometry as well as the material.

Importance of the stiffness matrix - For structural analysis, stiffness is a very important property. The equation for linear static analysis is [F] = [K] [D]. The force is usually known, the displacement is unknown, and the stiffness is a characteristic property of the element. This means if we formulate the stiffness matrix for a given shape, like line, quadrilateral, or tetrahedron, then the analysis of any geometry could be performed by meshing it and then solving the equation F = K D. Methods for formulating the stiffness matrix –

1. Direct Method
2. Variational Method
3. Weighted Residual Method

The direct method is easy to understand but difficult to formulate using computer programming. While the Variational and Weighted Residual Methods are difficult to understand, but easy from a programming point of view. That's the reason why all software codes either use the Variational or Weighted Residual Method formulation.

Methods for Formulating Stiffness Matrix

1. Variational Methods - Rayleigh Ritz method.
2. Weighted Residual methods -

   Galerkin Method,Sub-Domain Method,Collocation Method,Least Squares method.

   These methods are old approximate classical methods which were in use even before the advent of FEM

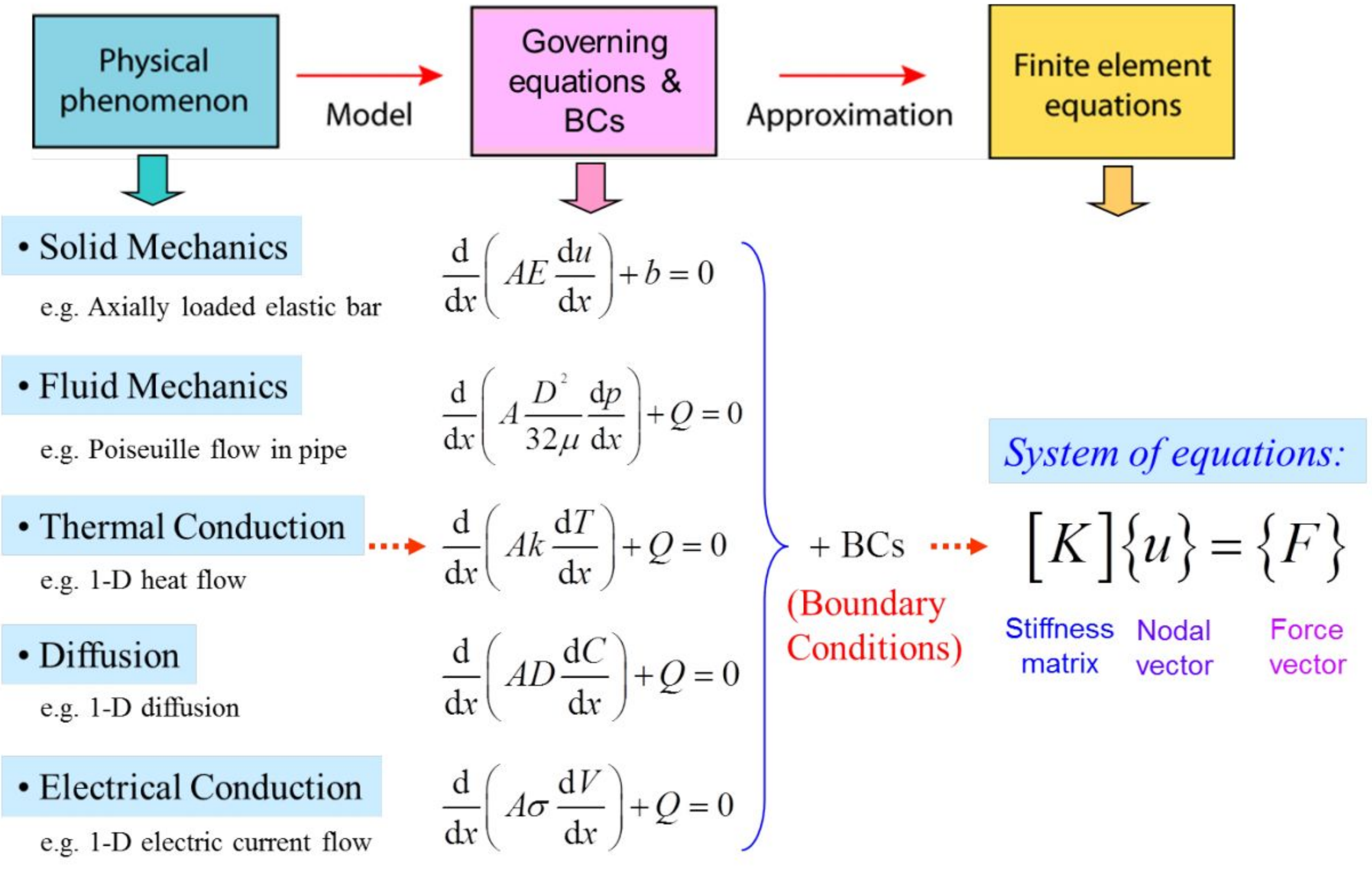
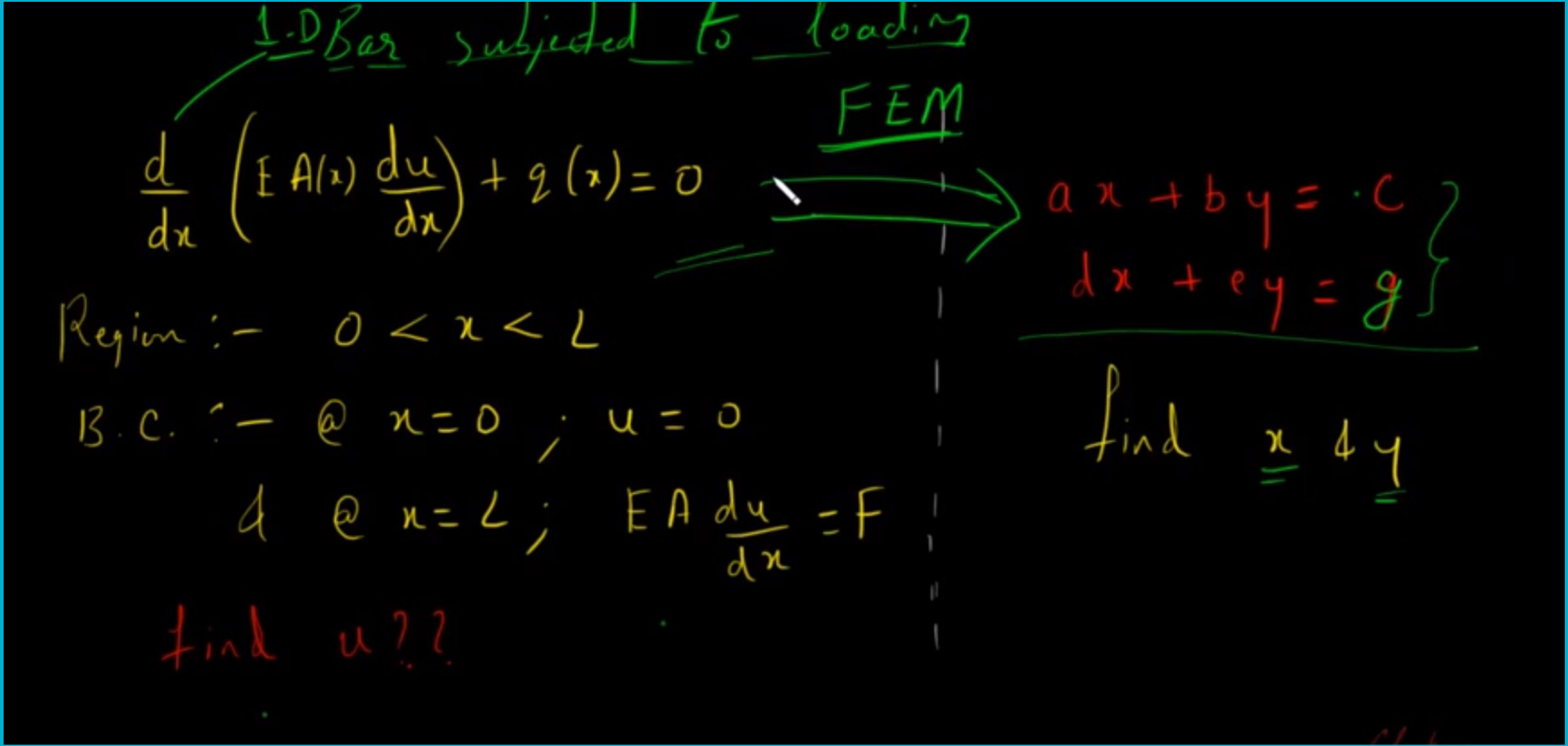
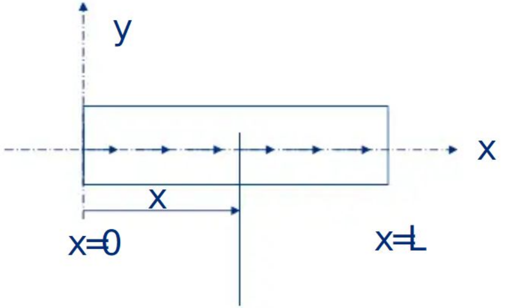


Figure 1: Governing equations for various physical phenomena



Variational Methods or galerkin method convert the Differential equations to algebraic equations which can be easily solved using computer as Differential equations are difficult to solve using computer

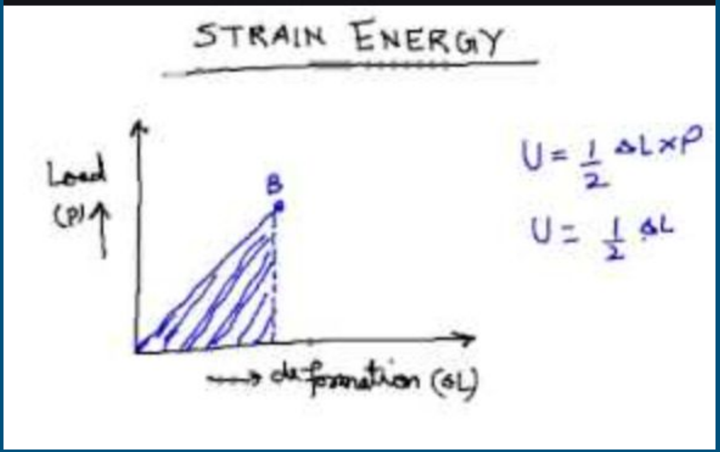
Axially loaded elastic bar



A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus
u(x) = displacement of the bar at x

Differential equation governing the response of the bar

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b=0; \qquad 0<x<L$$



Strain energy is a type of potential **energy** that is stored in a structural member as a result of elastic **deformation**. The external work done on such a member when it is deformed from its unstressed state is transformed into (and considered equal to the **strain energy** stored in it.

- The strain energy stored in the entire bar:

$$U=\int U^{*}dV=\int_0^L U^{*}A\,dx=\frac{1}{2}\int_0^L E\varepsilon_{xx}^2 A\,dx=\boxed{\frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 dx}$$

- Strain energy, U, for a uni-axial bar in extension

Variational Approach

In solving problems arising in physics and engineering it is often possible to replace the problem of integrating a differential equation by the equivalent problem of seeking a function that gives a minimum value of some integral. Problems of this type are called *variational problems*.

The methods that allow us to reduce the problem of integrating a differential equation to the equivalent variational problem are usually called *variational methods*.

Variational Approach

What is a functional?

functional → $$I(y) = \int_a^b F(x, y, y')dx$$

subjected to the boundary conditions

$$y(a) = A, \quad y(b) = B$$

Goal: Find a function F(x,y,y') for which the functional I(y) has an extremum (usually a minimum)

How to solve a problem using Variation calculus method

- Write the governing Differential equation of the engineering problem
- Convert the Differential equation to equivalent Integral form by comparing with Euler lagrange equation
- Assume a trial function in polynomial or trigonometric from.
- Trial function should satisfy the boundary conditions of the problem.
- Substitute the trial function in the equivalent Integral form.
- Minimize the Integral form of the functional wrt the unknown constants.
- From the above step get the value of unknown constants and substitute it in the trial solution to get the approximate solution.
- Get the values of strain and stress from the above steps.

Note: This is the classical procedure of solving without using FEM

Variational Approach

Question: Are there situations in which the function F(x,y,y') for which the functional I(y) in minimized is ALSO a solution to the PDE and BCs??

Answer: Yes, all PDEs typically found in physics and engineering have functionals or variational equations whose solution is equivalent to solving the PDE directly.

1.5 CALCULUS OF VARIATIONS

It is a method of finding maximum and minimum *or* stationary values of functional. A functional can be defined as function of several other functions. For example: Potential energy plays the role of the functional.

Consider a functional expressed as

$$A = \int_{x_1}^{x_2} F(x, u, u', u'')dx \qquad \text{---- (1.26)}$$

Where variable u and its first and second derivatives with respect to the independent variable x, u' and u'' are functions of x. Therefore, A and F are the functional.

Let this integral is defined in the region $[x_1 \;\; x_2]$ as shown in figure (1.6)

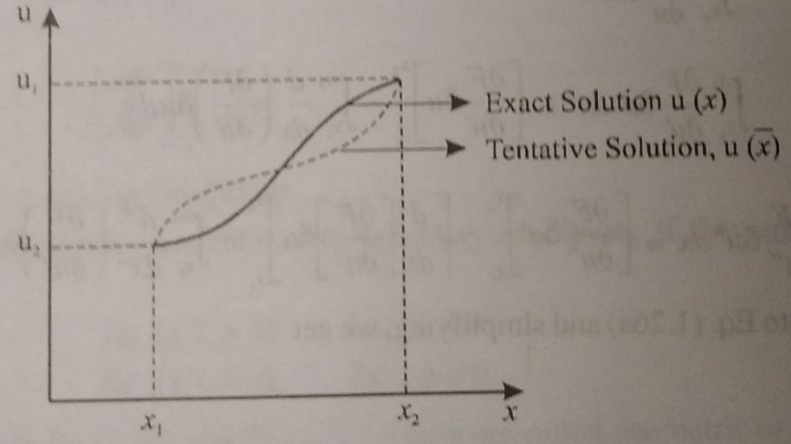
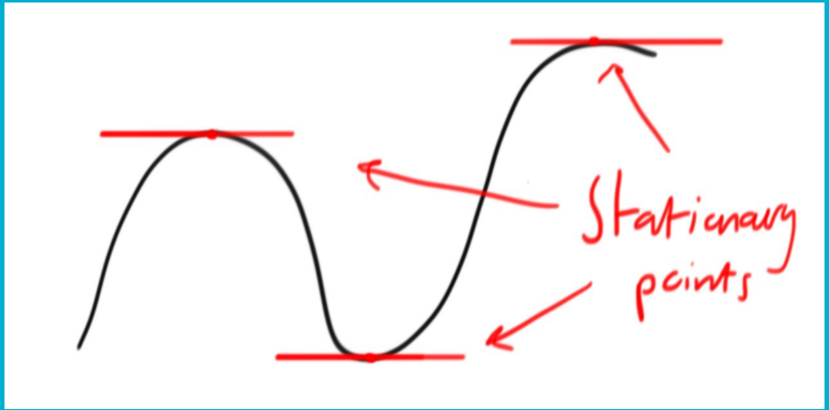
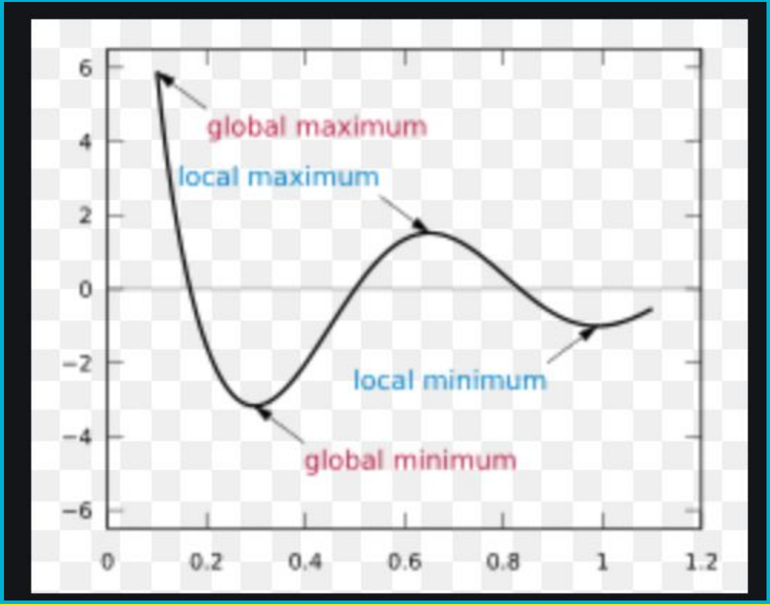


Fig. 1.6 : Region [x_1 x_2]





As we can see from this image, a stationary point is a point on a curve where the slop is zero

Hence the stationary points are when the derivative is zero

Hence to find the stationary point of $y = f(x)$, find $\frac{dy}{dx}$ and then set it equal to zero

$$\Rightarrow \frac{dy}{dx} = 0$$

Then solve this equation, to find the values of x for what the function is stationary

For examples

$$y = x^2 + 3x + 8$$

To find the stationary find $\frac{dy}{dx}$

$$\frac{dy}{dx} = 2x + 3$$

Set it to zero

$$2x + 3 = 0$$

Solve

$$x = -\frac{3}{2} \Rightarrow y = \frac{23}{4}$$

Hence the stationary point of this function is at $\left(-\frac{3}{2}, \frac{23}{4}\right)$

1.8 TOTAL POTENTIAL ENERGY (Π)

The total potential energy of an elastic body is defined as the sum of the strain energy due to internal stresses produced and the work potential due to the external force.

$$\therefore \text{Total potential energy(PE)} = \text{Strain Energy (SE)} + \text{Work Potential (WP)}$$

i.e. $$\text{PE functional, } \Pi = SE + WP \qquad \text{---- (1.35)}$$

1.8.1 Potential Energy Functional For a Three Dimensional Body

Consider a three-dimensional elastic body of volume v, subjected to body force, surface force and a point loads. Let u, v, *and* w be the displacement components in x, y, z direction respectively.

From Eq.(1.35), we have,

$$\text{Potential Energy Functional} = SE + WP$$

i.e. $$\Pi = SE + WP \qquad \text{---- (1.35a)}$$

The strain energy of the body is given by area under the curve (Figure 1.8) (For linear elastic materials)

The total **potential energy** of an elastic body , is defined as the sum of total strain **energy** (U) and the work **potential** (WP) .

1.6 EULER-LAGRANGE'S EQUATION

Let $A = \int_{x_1}^{x_2} F(x, u, u', u'')dx$ be the functional

where variable u and its first and second derivatives with respect to the variable x, u' and u'' are functions of x. and $[x_1, x_2]$ is the region in which 'A' is defined.

The condition for the functional A to be minimum is the variation is functional must be equal to *zero.*

i.e. $\delta A = 0$

Thus, from maximization *or* minimization of simple function in calculus

$$\delta A = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial u''}\delta u'' \right) dx = 0 \qquad \text{---- (1.26a)}$$

or

$$\delta A = \int_{x_1}^{x_2} \delta F\, dx = 0 \qquad \text{---- (1.26b}$$

Integration by parts for the respective terms in Eq.(1.26a) gives

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial u}\delta u dx \qquad \text{---- (1.2}$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial u'}\delta u'\, dx = \left[\frac{\partial F}{\partial u'}\delta u\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\delta u\, dx \qquad \text{---- (1.2}$$

and $$\int_{x_1}^{x_2} \frac{\partial F}{\partial u''}\delta u''\, dx = \left[\frac{\partial F}{\partial u''}\delta u'\right]_{x_1}^{x_2} - \left[\frac{d}{dx}\left[\frac{\partial F}{\partial u''}\right]\delta u\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''}\right)\delta u\, dx \qquad \text{---- (1.2}$$

Substituting into Eq. (1.26a) and simplifying, we get

$$\left.\begin{aligned} &\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''}\right)\right]\delta u dx \\ &+\left[\left(\frac{\partial F}{\partial u'} - \frac{d}{dx}\left(\frac{\partial F}{\partial u''}\right)\right)\delta u\right]_{x_1}^{x_2} \\ &+\left[\frac{\partial F}{\partial u''}\delta u'\right]_{x_1}^{x_2} = 0 \end{aligned}\right\} \qquad \text{---- (1.28}$$

Since δu is arbitrary, each term must vanish individually so that

$$\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''}\right)\right] = 0 \qquad \text{---- (1.2}$$

The Eq.(1.29) is know as the Euler equation or the Euler Lagrange's equation

In the calculus of variations and classical mechanics, the **Euler-Lagrange equations**[1] is a system of second-order ordinary differential equations whose solutions are stationary points of the given action functional. The equations were discovered in the 1750s by Swiss mathematician Leonhard Euler and Italian mathematician Joseph-Louis Lagrange.

1.8.2 Principle of Minimum Potential Energy

It states that "*Of all the displacement configuration a body can assume which satisfy compatibility conditions and boundary conditions, the configuration which satisfying equilibrium condition is one which will have the minimum potential energy*"

Thus, from Eq.(1.35),

Potential Energy functional, $\Pi = SE + WP$

For PE functional to be minimum,

$$\delta(\pi) = \delta(SE) + \delta(WP) = 0 \quad \text{---- (1.42)}$$

For example: Consider two bodies X and Y having the same weight falling through the different heights h_1 and h_2 as shown in figure (1.9), satisfies boundary conditions B and compatibility conditions C. Out of these two bodies, the body which has minimum potential energy is body Y because it attain the equilibrium state earlier than the body X, when allowed to fall freely. Thus any body which has the minimum potential energy satisfies stability condition or any body which is in stable state will have minimum potential energy.

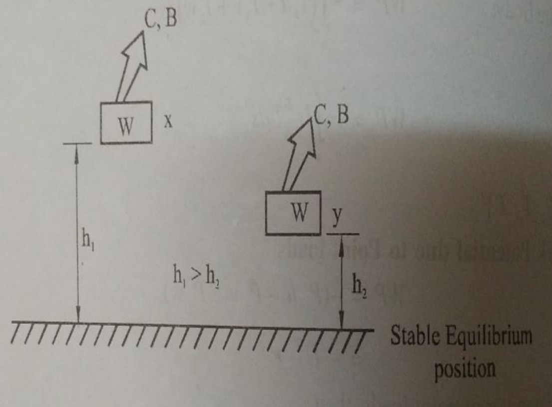


Fig. 1.9 : Equilibrium state

The **minimum total potential energy principle** is a fundamental concept used in physics and engineering. It dictates that at low temperatures a structure or body shall deform or displace to a position that (locally) minimizes the total potential energy, with the lost potential energy being converted into kinetic energy (specifically heat).

$$\frac{dF}{du} - \frac{d}{dx}\left(\frac{dF}{du'}\right) + \frac{d^2}{dx^2}\left(\frac{dF}{du''}\right) = 0$$

$$\frac{d}{dx}\left(\frac{dF}{du'}\right) - \frac{dF}{du} = 0 \Longrightarrow AE\frac{d}{dx}\left(\frac{du}{dx}\right) + q = 0$$

$$\frac{d}{dx}\left(\frac{dF}{du'}\right) = AE\frac{d}{dx}\left(\frac{du}{dx}\right) \; ; \; -\frac{dF}{du} = q$$

$$\frac{dF}{du'} = AE\,u'$$

$$F = \frac{AE}{2}(u')^2$$

$$F = -qu$$

$$I = \int_{x_1}^{x_2} F\,dx$$

Compare Governing DE to Euler Lagrange Equation

$$\frac{d}{dx}\left(\frac{dF}{du'}\right) - \frac{dF}{du} = 0 \Longrightarrow AE\frac{d}{dx}\left(\frac{du}{dx}\right) + q = 0$$

$$\frac{d}{dx}\left(\frac{dF}{du'}\right) = AE\frac{d}{dx}\left(\frac{du}{dx}\right) \; ; \; -\frac{dF}{du} = q$$

$$I = \int_{x_1}^{x_2} F\,dx$$

$$I = \int_0^l \left(\frac{AE}{2}\left(\frac{du}{dx}\right)^2 - qu\right) dx$$

Get Integral form of DE which is a functional. Here I integral is sum of SE + WP

In order to solve the governing differential equation using variation calculus we have to compare the DE with Euler Lagrange Equation and get the equivalent Integral form.The equivalent Integral form for a structural problem is the Potential Energy Functional. After this we can take trial function and solve this Integral Functional using the Rayleigh Ritz method.

How to solve problem using Rayleigh Ritz method

- Write the Potential Energy functional of the engineering problem
- P.E functional = S.E + W.P (strain energy + work potential)
- Assume a trial function in polynomial or trigonometric from.
- Trial function should satisfy the boundary conditions of the problem.
- Substitute the trial function in the PE functional.
- Minimize the P.E functional wrt the unknown constants.
- From the above step get the value of unknown constants and substitute it in the trial solution to get the approximate solution.
- Get the values of strain and stress from the above steps.

Note: This is the classical procedure of solving without using FEM

Rayleigh Ritz Method

Example 4

By R-R method, for a bar of cross sectional area A elastic modulus E, subjected to uniaxial loading P, show that at a distance x from fixed end is $u = \left(\frac{P}{AE}\right)x$ and hence determine the end deflection and the stress to which the bar is subjected to.

Solution:

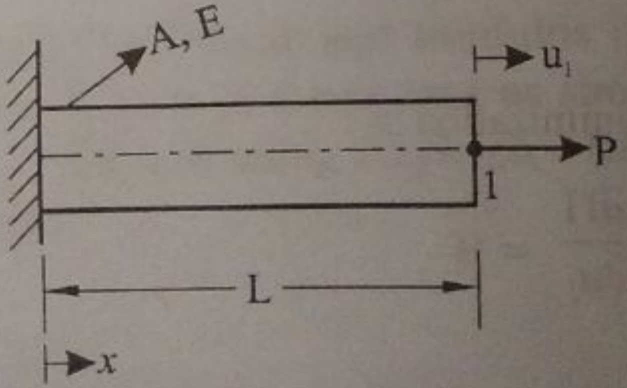


Fig. 1.12 : Axial loaded bar element

Let u be the axial displacement at any point x from the fixed end. When the bar subjected to uniaxial loading P at point 1. and let u_1 be the displacement at load point 1.

i) Formulate the PE functional

PE functional is given by $\Pi = SE + WP$

where Strain energy stored in bar is given by

$$SE = \frac{AE}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \text{ and}$$

Work potential is, $WP = -Pu_1$

$$\therefore \quad \Pi = \frac{AE}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx - Pu_1 \quad \text{---- (1}$$

ii) Assume a polynomial displacement function

$$u = a_0 + a_1 x \quad \text{---- (}$$

Where a_0 and a_1 are the generalized coordinates to be determined and the displacement functio should satisfy following boundary condition i.e.,

$$u = 0 \text{ at } x = 0; \quad \Rightarrow a_0 = 0$$

$\therefore$ From Eq.(2) $\quad u = a_1 x \quad$ ---- (

Differentiating Eq.(3) with respect to x, we get

$$\frac{\partial u}{\partial x} = a_1 \quad \text{---- (}$$

iii) Substitute the displacement function into PE functional

Substituting Eq.(3) and Eq.(4) into (1), we get

$$\Pi = \frac{AE}{2}\int_0^L a_1^2 dx - Pu_1$$

and $u = u_1$ when $x = L$ $\quad \therefore u_1 = a_1 L \quad$ from Eq.(3)

$$\therefore \quad \Pi = \frac{AE}{2} a_1^2 (x)_0^L - Pa_1 L$$

$$\Pi = \frac{AE}{2} a_1^2 L - Pa_1 L$$

iv) Minimize PE functional

The condition for the minimization is

$$\frac{\partial \Pi}{\partial a_i} = 0$$

From Eq.(5), we have $\quad \Pi = \frac{AE}{2} a_1^2 L - Pa_1 L$

$$\frac{\partial \Pi}{\partial a_1} = AE\,La_1 - PL = 0$$

$$\Rightarrow \quad a_1 = \frac{P}{AE}$$

v) Determination of displacement, strain and stress

To find Displacement

Thus, substituting value of a_1 into Eq.(3), we get

$$u = \left(\frac{P}{AE}\right)x$$

Hence the Proof.

To find end deflection and Stress

We know that, $x = L$ at the end

$$\therefore \quad u = \frac{P}{AE} \qquad \text{Ans.}$$

Therefore, from Eq.(6), we have

$$u = \left(\frac{P}{AE}\right)x$$

Thus, $\quad$ Strain, $\varepsilon = \frac{\partial u}{\partial x} = \frac{P}{AE}$

and from Hooke's law, we have

$$\sigma = \in E$$

i.e. $\quad \sigma = \frac{P}{AE}E$

$$\therefore \quad \text{Stress, } \sigma = \frac{P}{A} \text{ units} \qquad \text{Ans.}$$

Thank You